Analysis and Synthesis of Terminal Control Reduction Problem

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Abstract: *One of the most important problems of terminal control reduction problem is considered and solved. On the base of spinor representation of spatial rotation group, the control law functions are obtained. A mathematical model of reduction process is constructed. The results can be used for practical purposes to elaborate simple control algorithms of spatial movement reduction process in many different fields.*

Ketwords: Terminal Control, Moving Objects, Boundary Problems, Adaptive Control, Reduction, Acceleration, Transient Process.

Introduction

In the article [Erguven C. et al 2004], it has been shown that many motion control problems can be reduced to problems of control of terminal states of controlled objects. The main purpose of this investigation is to work out a simple adaptive method of terminal state control. One of the most important problems of terminal control is the problem of reduction of a moving object to the necessary state [Letov A.M.(1969)], which has wide practical applications.

Synthesis of the Control The reduction problem is defined by the following boundary conditions:

$$t = 0; \quad \gamma = \gamma_0; \quad \dot{\gamma} = \dot{\gamma}_0, \tag{1}$$

$$t = T; \quad \gamma = \gamma_f; \tag{2}$$

Conditions (1) and (2) mean that the object should be transferred from the initial state $\gamma = \gamma_0$ and $\dot{\gamma} = \dot{\gamma}_0$ to the state $\gamma = \gamma_f$ and at that its motion velocity should be arbitrary. In terms of variational calculus, this is a problem with moving ends. For this kind of problems, the given boundary conditions (1) and (2) are supplemented by the so-called natural boundary

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condition which in our case looks like [Kolmogorov A.N., Phomin C. (1972) Young L. (1974)].

$$G_{\dot{\gamma}} - \frac{d}{dt} G_{\ddot{\gamma}} = 0 \; , \tag{3}$$

where $G = k\ddot{\gamma}^2$.

Clearly, $G_{\dot{\gamma}} = 0$ and $G_{\ddot{\gamma}} = 2\ddot{\gamma}$. (4)

Condition (3) is reduced to the form

$$2\dddot{\gamma} = 0 \tag{5}$$

Differentiating

$$\gamma = C_0 + C_1 t + C_2 \frac{t^2}{2} + C_3 \frac{t^3}{6}. \tag{6}$$

thrice, taking into account the boundary conditions (1), (2) and the natural condition (5), we can define

$C_i \, (i = 0, 1, 2, 3)$ as follows:

$$C_3 = 0; C_2 = \frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\gamma_0}{T}; C_1 = \dot{\gamma}_0; C_0 = \gamma_0 \, . \tag{7}$$

Substituting (7) into the first and the second derivative of (6), we obtain the following expressions for an optimal trajectory in the phase space:

$$\gamma = \left(\frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T}\right)\frac{t^2}{2} + \dot{\gamma}_0 t + \gamma_0 \; , \tag{8}$$

$$\dot{\gamma} = \left(\frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T}\right)t + \dot{\gamma}_0 \; . \tag{9}$$

The acceleration (the second derivative in (8)) takes the form

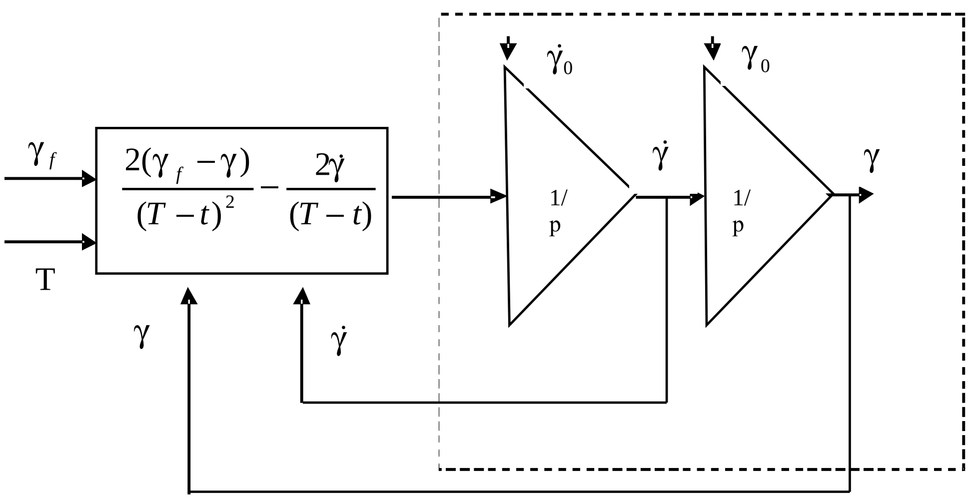
$$\ddot{\gamma} = \frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T} \; . \tag{10}$$

This is the law of control for the reduction problem. It means that if the acceleration of the controlled object on the time interval [0;T] is assumed to be constant and equal to (10), then at the moment of time $t = T$ its state will satisfy the boundary conditions $t = T$; $\gamma = \gamma_f$, $\dot{\gamma} = \dot{\gamma}_f$. However this is an open (program) law of control, i.e. the control law without feedback. Due to the possibility of direct measurements of the acceleration of a controlled object, (9) can be transformed to the control law with feedback. For this it suffices to assume the initial phase state to be the current one, i.e. to assume $\gamma = \gamma_0 \, u \dot{\gamma} = \dot{\gamma}_0$. In that case, the task fulfillment time should be assumed to be equal to the remaining time *T—t*. Then (10) takes the form

$$\ddot{\gamma} = \frac{2(\gamma_f - \gamma)}{(T-t)^2} - \frac{2\dot{\gamma}}{(T-t)} \qquad (11)$$

From (11), we see that in this case the acceleration acting on the controlled object stops to be constant and becomes dependent on the current velocity and coordinate values of the controlled object, i.e. , we have the realization of control with feedback. The block-diagram of the realization of control with feedback is presented in Figure 1. The measured coordinates of the current state $(\gamma ; \dot{\gamma})$ are delivered to the automatic control unit (ACU), where the required value of the influencing acceleration (11) is computed.

Figure 1: The block-diagram of the ACU of the reduction problem.



Analysis of the Control Process Dynamics in the Reduction Problem

It is clear that the motion program (8) (open control) is accelerated motion with constant acceleration (10). As has already been said, the transition to the control with feedback (11) transforms it to motion with variable acceleration. However, in that case, for *t* = *T* there arises one singularity – the denominator of the controlling function becomes equal to zero. This difficulty can be overcome by doing the following.

Assume that $T - t = \Delta T$, where ΔT is a constant time interval. From the physical standpoint this means that the target point of the reduction process is also moving, since it leaves the controlled object behind by the value ΔT. Denote its variable coordinate by γ_m. The controlling acceleration function on the time interval ΔT takes the form

$$\ddot{\gamma} = \frac{2(\gamma_m - \gamma)}{\Delta T^2} - \frac{2\dot{\gamma}}{\Delta T} \qquad (12)$$

where γ *and* $\dot{\gamma}$ (the coordinate and velocity of the controlled object) are, as previously, the variable values which are functions of time.

Thus when using the left-hand side of expression (12) for the controlling acceleration, it is assumed that the object moves with a constant lag in time by the value ΔT from the target point γ_m and and, after time T, its coordinate becomes equal to the given value $\gamma = \gamma_f$.

Now let us verify that this is really so. We begin by noting that by analogy with (8) we have the following program for the control of the coordinate of the moving target point

$$\gamma_m = \left(\frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T}\right)\frac{(t+\Delta T)^2}{2} + \dot{\gamma}_0(t+\Delta T) + \gamma_0, \qquad (13)$$

This equation reflects the fact that the leading point γ_m leaves the controlled object behind by time ΔT.

Substituting (13) into (12) and performing some simple transformations, we obtain the following expression for the controlling acceleration

$$\ddot{\gamma} = k_0 + k_1 t + k_2 t^2 + k_\gamma^1 \gamma + k_\omega^1 \omega \;, \qquad (14)$$

where $\omega = \dot{\gamma}$ is the velocity of the controlled object

$$k_0 = \frac{2\gamma_f}{T^2} + \frac{T - \Delta T}{\Delta T \cdot T}\left(2\gamma_0 \frac{T + \Delta T}{\Delta T \cdot T} + 2\omega_0\right);$$

$$k_1 = \frac{2\omega_0}{\Delta T}\left(\frac{1}{\Delta T} - \frac{2}{T}\right) + \frac{4(\gamma_k - \gamma_0)}{\Delta T \cdot T^2};$$

$$k_2 = \frac{2(\gamma_k - \gamma_0)}{\Delta T \cdot T^2} - \frac{2\omega_0}{\Delta T \cdot T}; \quad k_\gamma^1 = -\frac{2}{\Delta T^2}; \quad k_\omega^1 = -\frac{2}{\Delta T}$$

Expression (14) is a linear non-homogeneous differential equation of second order with constant coefficients

$$\ddot{\gamma} + k_\omega \dot{\gamma} + k_\gamma \gamma = k_0 + k_1 t + k_2 t^2, \qquad (15)$$

where $k_\gamma = -k_\gamma^1 \; u \; k_\omega = -k_\omega^1$.

As is known, its solution consists of two parts: a general solution of the corresponding homogeneous equation and a particular solution of the non-homogeneous equation. The first of these solutions is the so-called transitional component and the second solution is a stationary component [Kolmogorov A.N., Phomin C. (1972) Andronov A.A., (1981)].

Let us first define a particular solution, i.e. a stationary component. It will be sought in the form of a polynomial of the same structure as the right-hand part

$$\dot{\gamma} = a_0 + a_1 t + a_2 t^2 \qquad (16)$$

Substituting (17) into (16) and equating the right-hand parts, where powers *t* are assumed to be equal, we get the following equations for the coefficients a_i (i=0, 1, 2):

$$2a_2 + k_\omega a_1 + k_\gamma a_0 = k_0; \;\; 2k_\omega a_2 + k_\gamma a_1 = k_1; k_\gamma a_2 = k_2. \qquad (17)$$

(17) can be solved as follows:

$$a_2 = \frac{k_2}{k_\lambda}; a_1 = \frac{1}{k_\gamma}(k_1 - 2\frac{k_\omega k_2}{k_\gamma});$$

$$a_0 = \frac{1}{k_\gamma}(k_0 - \frac{2k_2}{k_\gamma} - \frac{k_\omega}{k_\gamma}(k_1 - 2\frac{k_\omega k_2}{k_\gamma})). \qquad (18)$$

Substituting the coefficients k_i (i = 0, 1, 2) from (14) into expressions (18), we finally obtain

$$a_2 = \frac{(\gamma_f - \gamma_0)}{T^2} - \frac{\dot{\gamma}_0}{T}; \;\; a_1 = \dot{\gamma}_0; \;\; a_0 = \gamma_0. \qquad (19)$$

Hence a particular solution of the non-homogeneous equation (15) has the form

$$\gamma = \left(\frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T}\right)\frac{t^2}{2} + \dot{\gamma}_0 t + \gamma_0, \qquad (20)$$

which coincides with (8) and thus indeed satisfies the boundary conditions (1) and (2).

Now let define a solution of the homogeneous equation

$$\ddot{\gamma} + k_\omega \dot{\gamma} + k_\gamma \gamma = 0, \qquad (21)$$

i.e. a transitional function of the reduction process.

The characteristic equation (21) can be written in the form

$$\lambda^2 + k_\omega \lambda + k_\gamma = 0 \ , \qquad (22)$$

which, as is easily seen, has two complex-conjugate roots

$$\lambda_1 = -\frac{1}{\Delta T} + \frac{1}{\Delta T} i \ \text{ and } \ \lambda_1 = -\frac{1}{\Delta T} - \frac{1}{\Delta T} i, \qquad (23)$$

where $i = \sqrt{-1}$.

By virtue of (23), a general solution of the homogeneous equation (21) can be written in the form

$$\gamma(t) = e^{-\frac{1}{\Delta T}t}\left(C_1 \cos\frac{t}{\Delta T} + C_2 \sin\frac{t}{\Delta T}\right) \ , \qquad (24)$$

which leads to a general solution of the non-homogeneous equation (15) [Andronov A.A., (1981)]

$$\gamma(t) = e^{-\frac{1}{\Delta T}t}\left(C_1 \cos\frac{t}{\Delta T} + C_2 \sin\frac{t}{\Delta T}\right) + \left(\frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T}\right)\frac{t^2}{2} + \dot{\gamma}_0 t + \gamma_0 \ . \qquad (25)$$

We also need a derivative (25)

$$\dot{\gamma}(t) = \omega(t) = -\frac{1}{\Delta T} e^{-\frac{1}{\Delta T}t}\left(C_1 \cos\frac{t}{\Delta T} + C_2 \sin\frac{t}{\Delta T}\right) + e^{-\frac{1}{\Delta T}t}\left(\frac{1}{\Delta T} C_2 \cos\frac{t}{\Delta T} - \frac{1}{\Delta T} C_1 \sin\frac{t}{\Delta T}\right)$$

$$\left(\frac{2(\gamma_f - \gamma_0)}{T^2} - \frac{2\dot{\gamma}_0}{T}\right)t + \dot{\gamma}_0. \qquad (26)$$

The initial values of functions (25) and (26) defined according to the initial conditions

$$\mathrm{t}=0;\ \gamma=\gamma_{10};\ \dot{\gamma}=\dot{\gamma}_{10}, \tag{27}$$

allow us to define the constants C_1 and C_2

$$C_1=\gamma_{10}-\gamma_0 \ \text{ and } \ C_2=\Delta T(\dot{\gamma}_{10}-\gamma_0)-(\gamma_{10}-\gamma_0) \tag{28}$$

and thereby the final form of a solution of the differential equation (15)

$$\gamma(t)=e^{-\frac{1}{\Delta T}t}[(\gamma_{10}-\gamma_0)\cos\frac{t}{\Delta T}+(\Delta T(\dot{\gamma}_{10}-\gamma_0)-(\gamma_{10}-\gamma_0))\sin\frac{t}{\Delta T}]+$$
$$+[(\frac{2(\gamma_f-\gamma_0)}{T^2}-\frac{2\dot{\gamma}_0}{T})\frac{t^2}{2}+\dot{\gamma}_0 t+\gamma_0] \tag{29}$$

Here we should make a remark concerning the initial conditions (27), since they differ from the first of the boundary conditions (1). The matter is that the reduction process can be started for any initial values of the coordinate and velocity of the controlled object. It is not obligatory that these values be equal to the calculated values of the coordinate and velocity of the controlled object which are given preliminarily in (1). If values (1) and (17) are not equal, then there occurs a transitional process defined by the exponential summand in (29). Otherwise, the transitional component is absent and the reduction process has to be content with the forced component, i.e. with the second summand in (29). It should also be noted that the transitional component has a damping character and the value Ä*T* plays the role of a time constant: the larger it is, the slower the damping process is, and vice versa. Thus the value Ä*T* can so-to-say serve as a measure of «strictness» of reduction process control.

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